Identify the number as real, complex, or pure imaginary.

\(2i\)

The complex numbers are an extension of the real numbers. They include numbers of the form \(a + bi\) where \(a\) and \(b\) are real numbers.

Determine if \(2i\) is a complex number.

\(2i\) is a complex number because it can be expressed as \(0 + 2i\), where 0 and 2 are real numbers.

The complex numbers include pure imaginary numbers of the form \(a + bi\) where \(a = 0\) and \(b \neq 0\), as well as real numbers of the form \(a + bi\) where \(b = 0\).

Choose the correct description(s) of \(2i\): complex and pure imaginary.

Express in terms of \(i\).

\[\sqrt{-196}\]

First, write \(-196\) as \(-1\) times 196.

\[\sqrt{-196} = \sqrt{-1(196)}\]

Next, write \(\sqrt{-1(196)}\) as the product of two radicals.

\[\sqrt{-1(196)} = \sqrt{196} \cdot \sqrt{-1}\]

Finally, simplify each radical.

\[\sqrt{196}\sqrt{-1} = 14i\]

Write the number as a product of a real number and \(i\). Simplify all radical expressions.

\[\sqrt{-19}\]

First, write \(-19\) as \(-1\) times 19.
\[ \sqrt{-19} = \sqrt{(-1) \cdot 19} \]

Next, write \( \sqrt{(-1) \cdot 19} \) as the product of two radicals.

\[ \sqrt{(-1) \cdot 19} = \sqrt{-1} \cdot \sqrt{19} \]

Simplify \( \sqrt{-1} \).

\[ \sqrt{-1} \sqrt{19} = i \sqrt{19} \]

Express in terms of \( i \).

\[ \sqrt{-150} \]

(Work out same as above.)

\[ 5i \sqrt{6} \]

Express in terms of \( i \).

\[ -\sqrt{-50} \]

\[ -\sqrt{-50} = -5i \sqrt{2} \]

Solve the equation.

\[ x^2 = -25 \]

Take the square root of both sides.

\[ x = \pm \sqrt{-25} \]

Rewrite the square root of the negative number.

\[ x = \pm i \sqrt{25} \]

Simplify: \( x = \pm 5i \)

The solutions are \( x = 5i, -5i \).

Solve the quadratic equation, and express all complex solutions in terms of \( i \).

\[ x^2 = 4x - 20 \]
First, write the equation in standard form.

\[ x^2 - 4x + 20 = 0 \]

Use the quadratic formula to solve.

\[ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \rightarrow \quad a = 1, \ b = -4, \ c = 20 \]

Substitute these values into the quadratic formula to get

\[ \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(20)}}{2(1)} \]

Simplify the radical expression.

\[ x = \frac{4 \pm 8i}{2} \quad \rightarrow \quad 2 + 4i, \ 2 - 4i. \]

Multiply.

\[ \sqrt{-8} \cdot \sqrt{-8} \]

Recall that \( \sqrt{-8} = i\sqrt{8} \).

First, rewrite \(-8\) as \(-1 \cdot 8\).

\[ \sqrt{-8} \cdot \sqrt{-8} = \sqrt{-1(8)} \cdot \sqrt{-1(8)} \]

Split into several radicals.

\[ \sqrt{-1(8)} \cdot \sqrt{-1(8)} = \sqrt{-1 \cdot \sqrt{8}} \cdot \sqrt{-1 \cdot \sqrt{8}} \]

Simply each radical, if possible.

\[ \sqrt{-1} \cdot \sqrt{\sqrt{8}} \cdot \sqrt{-1} \cdot \sqrt{\sqrt{8}} = (i)(\sqrt{8})(i)(\sqrt{8}) \]

Multiply.

\[ (i)(\sqrt{8})(i)(\sqrt{8}) = i^2(\sqrt{8} \cdot \sqrt{8}) = -8 \]

Divide.

\[ \frac{\sqrt{-192}}{\sqrt{-64}} \]
\[
\frac{\sqrt{-192}}{\sqrt{-64}} = \sqrt{3}
\]

Divide.

\[
\frac{\sqrt{-175}}{\sqrt{7}}
\]

Notice that the expression in the numerator is imaginary.

\[
\sqrt{-175} = i\sqrt{175}
\]

Then simplify the radical.

\[
i\sqrt{175} = 5i\sqrt{7} \rightarrow \frac{5i\sqrt{7}}{\sqrt{7}}
\]

Reduce the fraction to lowest terms by dividing out the common factor, \( \sqrt{7} \).

\[
\frac{5i\sqrt{7}}{\sqrt{7}} = 5i
\]

Add and simplify.

\[(7 + 5i) + (2 - 4i)\]

\[(7 + 5i) + (2 - 4i) = 9 + i\]

Multiply.

\[(7 + 8i)(5 + i)\]

\[(7 + 8i)(5 + i) = 47i + 27\]

Multiply.

\[(-6 + 9i)^2\]

\[(-6 + 9i)^2 = -108i - 45\]

Multiply.
\[
(\sqrt{10} + i)(\sqrt{10} - i) = 11
\]

Simplify.

\[i^{17}\]

Since 17 is odd, rewrite 17 as 16 + 1 and simplify.

\[i^{17} = i^{16+1} = i^{16} \cdot i\]

Write 16 as 2(8).

\[i^{16} \cdot i = i^{2(8)} \cdot i\]

Write \(i^{2(8)}\) as power of \(i^2\).

\[i^{2(8)} \cdot i = (i^2)^8 \cdot i\]

Remember that \(i^2 = -1\) and simplify.

\[(i^2)^8 \cdot i = (-1)^8 \cdot i\]

Remember that a negative number raised to an even power is positive.

\[(-1)^8 \cdot i = 1 \cdot i = i\]

Simplify.

\[i^{34}\]

\[i^{34} = -1\]

Simplify.

\[i^{15}\]

\[i^{15} = -i\]

Find the power of \(i\).
Use the rule for negative exponents.

\[ a^{-m} = \frac{1}{a^m} \]

\[ i^{-17} = \frac{1}{i^{17}} \]

Because \( i^2 = -1 \) is defined to be \(-1\), higher powers of \( i \) can be found. Larger powers of \( i \) can be simplified by using the fact that \( i^4 = 1 \).

\[
\frac{1}{i^{17}} = \frac{1}{i^4 \cdot i^4 \cdot i^4 \cdot i} \]

Since \( i^4 = 1 \), \( \frac{1}{i^{17}} = \frac{1}{i} \)

To simplify this quotient, multiply both the numerator and denominator by \(-i\), the conjugate of \( i \).

\[
\frac{1}{i} = \frac{1(-i)}{i(-i)} \rightarrow \frac{-i}{-i^2} \rightarrow \frac{-i}{-(-1)} = i
\]

Divide.

\[
\frac{8 + 4i}{8 - 4i}
\]

Reduce.

\[
\frac{2 + i}{2 - i}
\]

Multiply by a form of 1 determined by the conjugate of the denominator.

\[
\frac{2 + i}{2 - i} = \frac{2 + i}{2 - i} \cdot \frac{2 + i}{2 + i} = \frac{4 + 4i + i^2}{4 - i^2}
\]

Remember that \( i^2 = -1 \) and simplify.

\[
\frac{4 + 4i + i^2}{4 - i^2} = \frac{4 + 4i + (-1)}{4 - (-1)}
\]

\[
\frac{4 + 4i - 1}{4 + 1} = \frac{3 + 4i}{5}
\]
Write in the form $a + bi$.

$$\frac{3 + 4i}{5} = \frac{3}{5} + \frac{4}{5}i$$

---

**Trigonometric (Polar) Form of Complex Numbers**

The Complex Plane and Vector Representation ▪ Trigonometric (Polar) Form ▪ Converting Between Trigonometric and Polar Forms ▪ An Application of Complex Numbers to Fractals

Graph the complex number as a vector in the complex plane.

$-4 - 7i$

In the complex plane, the horizontal axis is called the real axis, and the vertical axis is called the imaginary axis.

The real part of the complex number is $-4$.

The imaginary part of the complex number is $-7$.

Graph the ordered pair $(-4, -7)$.

Draw an arrow from the origin to the point plotted.

Write the complex number in rectangular form.

$18(\cos 180° + i \sin 180°)$

Rewrite the equation in the form $a + bi$.

$18(\cos 180° + i \sin 180°) = 18 \cos 180° + (18 \sin 180°)i$

$a = 18 \cos 180°$

$b = 18 \sin 180°$

Simplify.

$a = 18 \cos 180°$

$a = (18)(-1)$

$a = -18$

Simplify.

$b = 18 \sin 180°$

$b = 18 \cdot 0$
\[ b = 0 \]

\[ 18(\cos 180° + i \sin 180°) = -18 + 0i = -18 \]

Write the complex number in rectangular form.

\[ 8(\cos(30°) + i \sin(30°)) \]

\[ 4\sqrt{3} + 4i \]

Write the complex number in rectangular form.

\[ 14 \text{ cis } 315° \]

Rewrite the equation.

\[ 14 \text{ cis } 315° = 14(\cos 315° + i \sin 315°) \]

\[ 14(\cos 315° + i \sin 315°) = 14 \cos 315° + (14 \sin 315°)i \]

\[ a = 14 \cos 315° \]
\[ b = 14 \sin 315° \]

Simplify.

\[ a = 14 \cos 315° \]
\[ a = 14 \cdot \frac{\sqrt{2}}{2} \]
\[ a = 7\sqrt{2} \]

Simplify.

\[ b = 14 \sin 315° \]
\[ b = 14 \cdot \frac{-\sqrt{2}}{2} \]
\[ b = -7\sqrt{2} \]

\[ 14 \text{ cis } 315° = 7\sqrt{2} - 7\sqrt{2}i \]

Find trigonometric notation.

\[ 5 - 5i \]
From the definition of trigonometric form of a complex number, you know that $5 - 5i = r(\cos \theta + i \sin \theta)$. Your goal is to find $r$ and $\theta$.

The definition says $r = \sqrt{a^2 + b^2}$ for a complex number $a + bi$. So, for the complex number $5 - 5i$, $r = \sqrt{(5)^2 + (-5)^2}$.

In its reduced form, $\sqrt{(5)^2 + (-5)^2} = 5\sqrt{2}$.

Use the value of $r$ to find $\theta$. For a complex number $a + bi$, $a = r \cos \theta$. For the complex number $5 - 5i$, $5 = r \cos \theta$ and $r = 5\sqrt{2}$, $5 = 5\sqrt{2} \cos \theta$.

Solve the equation for $\cos \theta$.

$\cos \theta = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Using the equation $b = r \sin \theta$ and solving for $\sin \theta$, you get

$\sin \theta = \frac{-5}{5\sqrt{2}} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$.

Placing the angle in the same quadrant as $5 - 5i$, $\theta$, written as a degree measure, is $315^\circ$.

$5 - 5i = 5\sqrt{2}(\cos 315^\circ + i \sin 315^\circ)$

Write the complex number $-2 - 2i$ in trigonometric form $r(\cos \theta + i \sin \theta)$, with $\theta$ in the interval $[0^\circ, 360^\circ]$.

$-2 - 2i = \text{Find } r \text{ by using the equation } r = \sqrt{x^2 + y^2} \text{ for the number } xy i$.

$r = \sqrt{(-2)^2 + (-2)^2} \rightarrow 2\sqrt{2}$

Use the value of $r$ to find $\theta$. For a complex number $x + yi$, $x = r \cos \theta$.

$-2 = 2\sqrt{2} \cos \theta$.

Solve the equation for $\cos \theta$.

$\cos \theta = \frac{-2}{2\sqrt{2}} \rightarrow = \frac{-1}{\sqrt{2}}$

Solve for $\theta$. 

\[ \theta = \cos^{-1}\left( -\frac{1}{\sqrt{2}} \right) \]

Placing the angle in the same quadrant as \(-5 - 5i\), \(\theta\), written in degree measure, is 225°.

\[-2-2i = 2\sqrt{2}(\cos 225^\circ + i \sin 225^\circ)\]

Find the trigonometric form. \(0 \leq \theta < 360\)

3

From the definition of trigonometric form of a complex number, you know that \(3 = r(\cos \theta + i \sin \theta)\). Your goal is to find \(r\) and \(\theta\).

The definition says \(r = \sqrt{a^2 + b^2}\) for a complex number \(a + bi\). So, for the complex number 3, \(r = \sqrt{(3)^2 + (0)^2}\).

In its reduced form, \(\sqrt{(3)^2 + (0)^2} = 3\).

Use the value of \(r\) to find \(\theta\). For a complex number \(a + bi\), \(a = r \cos \theta\). For the complex number 3, \(3 = r \cos \theta\) and \(r = 3\), so \(3 = 3 \cos \theta\). Solve the equation for \(\cos \theta\).

\[ \cos \theta = \frac{3}{3} = 1 \]

\[ \theta = \cos^{-1}(1) \]

\(\theta\), written as a degree measure, is 0°.

\(3 = 3 (\cos 0^\circ + i \sin 0^\circ)\)

Write the complex number 6 \((\cos 98^\circ + i \sin 98^\circ)\) in the form \(a + bi\).

To solve this problem, first find the cosine of 98° and the sine of 98°.

The cosine of 98° is \(-0.139173\), rounded to six decimal places.

The sine of 98° is \(0.990268\), rounded to six decimal places.

\[ 6(\cos 98^\circ + i \sin 98^\circ) = 6(-0.139173 + i 0.990268) \quad \rightarrow \]

\[ 6 \cdot -0.139173 + i 6 \cdot 0.990268 \]

So, the answer in \(a + bi\) form is \(-0.8 + 5.9i\).
Convert the rectangular form $9i$ into trigonometric form.

To find $r$, use the formula $r = \sqrt{x^2 + y^2}$.

In this case, $x = 0$ and $y = 9$.

$$r = \sqrt{(0)^2 + (9)^2} \rightarrow r = \sqrt{81} \rightarrow r = 9$$

Sketch the graph of $9i$. Note that because $x$ is 0, $\theta$ is a quadrant angle.

$\theta = 90^\circ$

The trigonometric form of the complex number $x + yi$ is $r(\cos \theta + i \sin \theta)$. Substitute the values of $r$ and $\theta$ into this form to find the trigonometric form of $9i$.

$9i = 9(\cos 90^\circ + i \sin 90^\circ)$

---

Convert the rectangular form $-3 + 5i$ into trigonometric form.

$r = \sqrt{x^2 + y^2}$

In this case, $x = -3$ and $y = 5$.

$$r = \sqrt{(-3)^2 + (5)^2} \rightarrow r = \sqrt{34} \rightarrow r \approx 5.8$$

To find the degree measure of the and $\theta$, use the following formula.

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ if $x > 0$ or $\theta = 180 + \tan^{-1}\left(\frac{y}{x}\right)$ if $x < 0$.

Since $x < 0$, use the second formula.

$$\tan^{-1}\left(\frac{5}{-3}\right) = \tan^{-1}\left(\frac{-5}{3}\right) \approx -59.0363^\circ \rightarrow \theta \approx -59.0363^\circ$$

Add 180 to get positive angle.

$\theta \approx 121.0^\circ$, rounded to nearest tenth.

$-3 + 5i = 5.8(\cos 121.0^\circ + i \sin 121.0^\circ)$

---

**The Product and Quotient Theorems**

**Products of Complex Numbers in Trigonometric Form**
Quotients of Complex Numbers in Trigonometric Form

Multiply the two complex numbers.

\[3(\cos 146° + i \sin 146°) \cdot 7(\cos 102° + i \sin 102°)\]

\[3(\cos 146° + i \sin 146°) \cdot 7(\cos 102° + i \sin 102°) = 21 \cos 248° + i \sin 248°\]

Convert your answer to a + bi form.

\[21(\cos 248° + i \sin 248°) = 21 (– 0.3746066 + i (– 0.92718385)) = – 7.8667386 + (– 19.47086085)i\]

Rounding each number to the nearest tenth gives the following.

\[3(\cos 146° + i \sin 146°) \cdot 7(\cos 102° + i \sin 102°) = – 7.9 – 19.5i\]

Find the following product, and write the product in rectangular form, using exact values.

\[2(\cos 60° + i \sin 60°)[5(\cos 180° + i \sin 180°)]\]

To multiply two complex numbers, use the fact that if they are expressed in polar form \( r_1(\cos \theta_1 + i \sin \theta_1) \) and \( r_2(\cos \theta_2 + i \sin \theta_2) \), there product is \( r_1r_2(\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)) \).

Multiply these two complex numbers.

\[2(\cos 60° + i \sin 60°)[5(\cos 180° + i \sin 180°)] = (2 \cdot 5)\cos (60° + 180°) + i \sin (60° + 180°)\]

= 10[cos(240°) + i sin(240°)]

Determine the exact values of cos 240° and sin 240°. Replace these into the expression and simplify the result.

\[= 10 (\frac{-1}{2} + i \frac{-\sqrt{3}}{2}) \rightarrow 10 (\frac{-1}{2}) + i (10)(\frac{-\sqrt{3}}{2})\]

\[= – 5 – 5\sqrt{3}i\]

Find the following product, and write the result in rectangular form using exact values.
(2 cis 45°)(5 cis 105°)

Rewrite.

(2 cis 45°)(5 cis 105°) = 

\[2(\cos 45° + i \sin 45°)][5(\cos 105° + i \sin 105°)]\]

To multiply two complex numbers, use the fact that if they are expressed in polar form 

\[r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad r_2(\cos \theta_2 + i \sin \theta_2)\]

there product is 

\[r_1r_2(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))\]

Multiply these two complex numbers.

\[2(\cos 45° + i \sin 45°)][5(\cos 105° + i \sin 105°)]\]

\[= (2 \cdot 5)[\cos(45° + 105°) + i \sin (45° + 105°)]\]

\[= 10[\cos(150°) + i \sin (150°)]\]

Notice that you know the exact values of \(\cos 150°\) and \(\sin 150°\). Replace these into the expression and simplify the result.

\[(2 \text{ cis } 45°)(5 \text{ cis } 105°) = 10\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \rightarrow \]

\[10\left(-\frac{\sqrt{3}}{2}\right) + (10)\left(\frac{1}{2}i\right) = -5\sqrt{3} + 5i\]

Find the quotient and write it in rectangular form using exact values.

\[\frac{20(\cos 120° + i \sin 120°)}{5(\cos 180° + i \sin 180°)}\]

To simplify the quotient, first use the fact that

\[\frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))\]

\[\frac{20(\cos 120° + i \sin 120°)}{5(\cos 180° + i \sin 180°)} = 4(\cos 60° + i \sin 60°)\]

Next, write this product in rectangular form by evaluating the cosine and sine functions.

\[\cos 60° = \frac{1}{2}\]

\[\sin 60° = \frac{\sqrt{3}}{2}\]
Expand the final product.

\[
\frac{20(\cos 120^\circ + i \sin 120^\circ)}{5(\cos 180^\circ + i \sin 180^\circ)} = 2 - 2\sqrt{3}i
\]

Use a calculator to perform the indicated operations.

\[
\frac{73(\cos 109^\circ + i \sin 109^\circ)}{36.5(\cos 69^\circ + i \sin 69^\circ)} = 1.53 + 1.29i \text{ (rounded to nearest hundredth)}
\]

Use a calculator to perform the indicated operation. Give the answer in rectangular form.

\[
(2 \text{cis} \frac{6\pi}{7})^2
\]

Use the fact that \(x^2 = x \cdot x\) to rewrite the problem.

\[
(2 \text{cis} \frac{6\pi}{7})^2 = (2 \text{cis} \frac{6\pi}{7})(2 \text{cis} \frac{6\pi}{7})
\]

Use the product rule.

\[
(2 \text{cis} \frac{6\pi}{7})(2 \text{cis} \frac{6\pi}{7}) = 4 \text{cis} \frac{12\pi}{7}
\]

Remember \(\text{cis} \theta = \cos \theta + i \sin \theta\). Rewrite the right side of the equation.

\[
4 \text{cis} \frac{12\pi}{7} = 4(\cos \frac{12\pi}{7} + i \sin \frac{12\pi}{7})
\]

Using a calculator, find the values of \(\cos \frac{12\pi}{7}\) and \(\sin \frac{12\pi}{7}\).

\[
4 \text{cis} \frac{12\pi}{7} = 4(0.623490 - 0.781831i)
\]

Multiply through by 4.

\[
4(0.623490 - 0.781831i) = 2.4940 - 3.1273i
\]

The alternating current in an electric inductor is \(I = \frac{E}{Z}\), where \(E\) is voltage and \(Z = R + X_L\) is impedance. If
To find the current, it is convenient to write all complex numbers in trigonometric form. \( E \) is already in trigonometric form, though you can write it more conveniently.

\[ E = 5 \text{ cis } 50^\circ \]

Convert \( Z \) into trigonometric form.

To convert from rectangular form \( x + yi \) to trigonometric form \( r \text{ cis } \theta \), remember \( r = \sqrt{x^2 + y^2} \) and \( \tan \theta = \frac{y}{x} \). Use these equations to solve for \( r \) and \( \theta \).

\[ Z = 8.0623 \text{ cis } 29.7449^\circ \] (rounded to four decimal places)

Substitute in the values for \( E \) and \( Z \), and then apply the quotient rule.

\[ I = \frac{E}{Z} = \frac{5 \text{ cis } 50^\circ}{8.0623 \text{ cis } 29.7449^\circ} \]

The quotient of two complex numbers in trigonometric form may be simplified by

\[
\frac{r_1 \text{ cis } \theta_1}{r_2 \text{ cis } \theta_2} = \frac{r_1}{r_2} \text{ cis } (\theta_1 - \theta_2).
\]

\[ = 0.6202 \text{ cis } 20.2551^\circ \] (rounded to four decimal places)

Convert back to rectangular form by expanding.

Remember that \( \text{cis } \theta \) means \( \cos \theta + i \sin \theta \). Use a calculator to evaluate the sine and cosine of \( 28.1986^\circ \).

\[ 0.6202 \text{ cis } 20.2551^\circ = 0.58 + 0.21i \]

---

**De Moivre’s Theorem: Powers and Roots of Complex Numbers**

**Powers of Complex Numbers (De Moivre’s Theorem)**

**Roots of Complex Numbers**

Find the following power. Write the answer in rectangular form.

\[ [5(\cos 30^\circ + i \sin 30^\circ)]^2 \]

To find the powers of a complex number, use De Moivre's theorem.

\[ [5(\cos 30^\circ + i \sin 30^\circ)]^2 = 5^2(\cos 2 \cdot 30^\circ + i \sin 2 \cdot 30^\circ) \]
\[ = 25(\cos 60° + i \sin 60°) \]

Find the exact values of \( \cos 60° \) and \( \sin 60° \) and simplify.

\[ = 25\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \]

In \( a + bi \) form: \[ \frac{25}{2} + \frac{25\sqrt{3}}{2}i \]

Raise the number to the given power and write the answer in rectangular form.

\[ \left(\sqrt{10}(\cis 120°)\right)^4 \]

Use De Moivre's theorem.

\[ [r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta) \]

The modulus \( \sqrt{10} \), raised to the power of 4, is 100.

The argument 120°, multiplied by 4, is 480°.

\[ \left[\sqrt{10}(\cis 120°)\right]^4 = 100(\cos 480° + i \sin 480°) \]

Use the distributive property.

100(\cos 480° + i \sin 480°) = 100 \cdot \cos 480° + 100 \cdot i \sin 480°

The product 100 \cdot \cos 480° is equal to –50.

The product of 100 \cdot \sin 480° is equal to \( 50\sqrt{3}i \).

Add.

\[ \left[\sqrt{10}(\cis 120°)\right]^4 = -50 + 50i\sqrt{3} \]

Find the given power.

\[ (4 + 4i)^7 \]

TI-83: \( (4 + 4i)^7 = 131072 - 131072i \).

Find the cube roots of 125(\cos(0°) + i \sin(0°)).
Find the cube roots of $125\left(\cos(0^\circ) + i \sin (0^\circ)\right)$.

The number of cube roots of $125\left(\cos(0^\circ) + i \sin (0^\circ)\right)$ is 3.

The roots of a complex number with modulus $r$ all have the same modulus, $\frac{1}{r^n}$. In this case, $r = 125$.

$$\frac{1}{r^n} = \left(\frac{1}{125}\right)^{\frac{1}{3}} = 5$$

The $n$th roots of a complex number are distinguished by $n$ different arguments. Find these arguments by using the following formula.

$$a = \frac{\theta + 2k \pi}{n} = \frac{\theta + k \cdot 360^\circ}{n} \quad \text{for } k = 0, 1, 2, \ldots, n-1.$$

In this case $n = 3$, $\theta = 0^\circ$, and $k = 0, 1, 2$.

When $k = 0$, $a = \frac{0^\circ + (0)(360^\circ)}{3} = 0^\circ$.

That gives the complex root $5\left(\cos 0^\circ + i \sin 0^\circ\right)$.

When $k = 1$, $a = \frac{0^\circ + (1)(360^\circ)}{3} = 120^\circ$.

This gives the complex root $5\left(\cos 120^\circ + i \sin 120^\circ\right)$.

When $k = 2$, $a = \frac{0^\circ + (2)(360^\circ)}{3} = 240^\circ$.

This gives the complex root $5\left(\cos 240^\circ + i \sin 240^\circ\right)$.

The complex roots are:

$5\left(\cos 0^\circ + i \sin 0^\circ\right)$,
$5\left(\cos 120^\circ + i \sin 120^\circ\right)$, and
$5\left(\cos 240^\circ + i \sin 240^\circ\right)$.

Find the cube roots of $27 \left(\cos 90^\circ + i \sin 90^\circ\right)$.

Find the cube roots of $27\left(\cos 90^\circ + i \sin 90^\circ\right)$. The number of cube roots($\cos 90^\circ + i \sin 90^\circ$) is 3.

The roots of a complex number with modulus $r$ all have the same modulus, $\frac{1}{r^n}$. In this case, $r = 27$.

$$\frac{1}{r^n} = \left(\frac{1}{27}\right)^{\frac{1}{3}} = 3$$
The $n$th roots of a complex number are distinguished by $n$ different arguments. Find these arguments by using the following formula.

$$a = \frac{\theta + 2k\pi}{n} = \frac{\theta + k \cdot 360^\circ}{n} \quad \text{for} \quad k = 0, 1, 2, \ldots, n - 1.$$

In this case $n = 3$, $\theta = 90^\circ$, and $k = 0, 1, \text{and } 2$.

When $k = 0$, \[a = \frac{90^\circ + (0)(360^\circ)}{3} = 30^\circ.\]
This gives the complex root $3(\cos 30^\circ + i \sin 30^\circ)$.

When $k = 0$, \[a = \frac{90^\circ + (1)(360^\circ)}{3} = 150^\circ.\]
This gives the complex root $3(\cos 150^\circ + i \sin 150^\circ)$.

When $k = 0$, \[a = \frac{90^\circ + (2)(360^\circ)}{3} = 270^\circ.\]
This gives the complex root $3(\cos 270^\circ + i \sin 270^\circ)$.

The complex roots are:
$3(\cos 30^\circ + i \sin 30^\circ)$,
$3(\cos 150^\circ + i \sin 150^\circ)$, and
$3(\cos 270^\circ + i \sin 270^\circ)$.

Find the cube roots of the following complex number. Express your answer in 'cis' form. Then plot the cube roots.

$-8$

To begin, convert $-8$ into the polar form of the same number.

To express $-8$ in polar form, begin by sketching the graph of $-8$ in the complex plane, then find the vector with direction $\theta$ and magnitude $r$ corresponding to the number in rectangular form.

Remember that $r = \sqrt{x^2 + y^2}$ and $\tan \theta = \frac{y}{x}$.

In the polar form, $-8 = 8 \text{cis } 180$.

Next, find the cube roots. These are the numbers of the form $a + bi$, where $(a + bi)^3$ is equal to $-8$. Suppose this complex number is $r(\cos \alpha + i \sin \alpha)$. Then you want the third power to be equal to $8(\cos 180^\circ + i \sin 180^\circ)$.

$$r^3(\cos 3\alpha + i \sin 3\alpha) = 8(\cos 180^\circ + i \sin 180^\circ)$$

Satisfy part of this equation by letter $r^3 = 8$. Solve this for $r$.

$r^3 = 8 \quad \rightarrow \quad r = 2$

Next, make $n\alpha$ an angle coterminal with $180^\circ$. Therefore, you must
have \[3 \alpha = 180° + 360° \cdot k\], where \(k\) is any nonnegative integer less than \(n\).

\[\alpha = 60° + 120° \cdot k\]

Let \(k\) take on integer values, starting with 0. If \(k = 0\), then \(\alpha = 60°\).

If \(k = 1\), then \(\alpha = 180°\).

If \(k = 2\), \(\alpha = 300°\). Notice that if \(k = 3\), \(\alpha = 60° + 360°\), which will be coterminal with \(60°\), so you would just be repeating solutions already found if higher values of \(k\) were used.

The solutions will be \(2 \text{ cis } 60°\), \(2 \text{ cis } 180°\), and \(2 \text{ cis } 300°\).

The three solutions can be graphed as shown below.

Find the cube roots of the following complex number. Then plot the cube roots.

\[-7\sqrt{3} + 7i\]

To begin, convert \(-7\sqrt{3} + 7i\) into the polar form of the same number.

To express \(-7\sqrt{3} + 7i\) in polar form, begin by sketching the graph of \(-7\sqrt{3} + 7i\) in the complex plane, then find the vector with direction \(\theta\) and magnitude \(r\) corresponding to the number in rectangular form. Remember that \(r = \sqrt{x^2 + y^2}\) and \(\tan \theta = \frac{y}{x}\).

In polar form, \(-7\sqrt{3} + 7i = 14 \text{ cis } 150°\).

Next, find the cube roots. These are the numbers of the form \(a + bi\), where \((a + bi)^3 = -7\sqrt{3} + 7i\). Suppose this complex number is \(r(\cos \alpha + i \sin \alpha)\). Then you want the third power to be equal to \(14 \text{ cis } 150°\).

\[r^3 (\cos 3\alpha + i \sin 3\alpha) = 14 (\cos 150° + i \sin 150°)\]
Satisfy part of this equation by letting $r^3 = 14$. Solve this for $r$.

$$r^3 = 14 \rightarrow \sqrt[3]{14}$$

Next, make $n\alpha$ an angle coterminal with $150^\circ$. Therefore, you must have $3\alpha = 150^\circ + 360^\circ \cdot k$, where $k$ is any nonnegative integer less than $n$.

$$\alpha = 50^\circ + 120^\circ \cdot k$$

Now let $k$ take on integer values, starting with 0.

If $k = 0$, then $\alpha = 50^\circ$.
If $k = 1$, then $\alpha = 170^\circ$.
If $k = 2$, then $\alpha = 290^\circ$. Notice the if $k = 3$, $\alpha = 50^\circ + 360^\circ$, which will be coterminal with $50^\circ$, so you would just be repeating solutions already found if higher values of $k$ were used.

The solutions will be $\sqrt[3]{14} \text{cis} 50^\circ$, $\sqrt[3]{14} \text{cis} 170^\circ$, and $\sqrt[3]{14} \text{cis} 290^\circ$.

Find all solutions to the equation.

$$x^3 + 512 = 0$$

Since $x^3 + 512 = 0$, $x^3 = -512$. 'Solving the equation' is equivalent to finding the cube roots of $-512$. Begin by writing $-512$ in trigonometric form.

In trigonometric form, the modulus is 512.

Find the smallest nonnegative value $\theta$ for which $-512 = 512 \cos \theta$.

The argument, written as a degree measure, is $180^\circ$. 

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![Graph showing solutions](image-url)
\[ -512 = 512(\cos 180^\circ + i \sin 180^\circ) \]
The number of cube roots of \(512(\cos 180^\circ + i \sin 180^\circ)\) is 3.

The roots of a complex number with modulus \(r\) all have the same modulus \(\frac{1}{r^n}\). In this case \(r = 512\).

\[
\frac{1}{r^n} = 512^{\frac{1}{3}} = 8
\]

Then \(n\) nth roots of a complex number are distinguished by \(n\) different arguments. You can find these arguments by using the following formula.

\[
a = \frac{\theta + 360 \cdot k}{n} \quad \text{for} \quad k = 0, 1, 2, \ldots, n-1.
\]

In this case, \(n = 3\), \(\theta = 180^\circ\), and \(k = 0, 1, \text{ and } 2\).

When \(k = 0\), \(a = \frac{180^\circ + (360^\circ)(0)}{3} = 60^\circ\).

This gives you the complex root \(8(\cos 60^\circ + i \sin 60^\circ)\).

When \(k = 0\), \(a = \frac{180^\circ + (360^\circ)(1)}{3} = 180^\circ\).

This gives you the complex root \(8(\cos 180^\circ + i \sin 180^\circ)\).

When \(k = 2\), \(a = \frac{180^\circ + (2)(360^\circ)}{3} = 300^\circ\).

This gives you the complex root \(8(\cos 300^\circ + i \sin 300^\circ)\).

The solutions to \(x^3 + 512 = 0\) are:
\(8(\cos 60^\circ + i \sin 60^\circ)\),
\(8(\cos 180^\circ + i \sin 180^\circ)\), and
\(8(\cos 300^\circ + i \sin 300^\circ)\).

Find all solutions to the equation \(x^3 + 64 = 0\).

Since \(x^3 + 64 = 0\), \(x^3 = -64\). “Solving the equation” is equivalent to finding the cube roots of \(-64\). Begin by writing \(-64\) in trigonometric form.

Find the modulus using \(\sqrt{0^2 + (-64)^2}\) and simplify the radical.

In trigonometric form, the modulus is 64.

You want to find the smallest nonnegative value \(\theta\) for which \(-64 = \)
64\cos \theta.

The argument, written as a degree measure, is 180°.

\[-64 = 64(\cos 180^\circ + i \sin 180^\circ).\]

The number of cube roots of \(64(\cos 180^\circ + i \sin 180^\circ)\) is 3.

The roots of a complex number with modulus \(r\) all have the same modulus \(\frac{1}{r^{\frac{1}{n}}}\).

In this case \(r = 64\), so \(\frac{1}{r^{\frac{1}{n}}} = \frac{1}{64^{\frac{1}{3}}} = 4\).

The \(n\) \(n\)th roots of a complex number are distinguished by \(n\) different arguments. You can find these arguments by using the formula

\[\alpha = \frac{\theta + 2k \pi}{n} = \frac{\theta + k \cdot 360^\circ}{n}\]

for \(k = 0, 1, 2, \ldots, n - 1\). In this case, \(n = 3, \theta = 180^\circ,\) and \(k = 0, 1,\) and 2.

When \(k = 0, \quad \alpha = \frac{180^\circ + (0)(360^\circ)}{3} = 60^\circ.\)

This gives the complex root \(4(\cos 60^\circ + i \sin 60^\circ)\).

\[= 4\left(\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}\right) = 2 + 2i \sqrt{3}\]

When \(k = 1, \quad \alpha = \frac{180^\circ + (1)(360^\circ)}{3} = 180^\circ.\)

\[= 4(-1 + i(0)) = -4\]

When \(k = 2, \quad \alpha = \frac{180^\circ + (2)(360^\circ)}{3} = 300^\circ.\)

This gives the complex root \(4(\cos 300^\circ + i \sin 300^\circ)\).

\[= 4\left(\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right) = 2 - 2i \sqrt{3}\]

Find all complex solutions.

\[x^4 + 10 = 0\]

To solve this equation, first isolate the power on one side of the equals sign, and take the fourth root.

\[x^4 + 10 = 0 \quad \rightarrow \quad x^4 = -10\]
To find the fourth root, first write $-10$ in polar form.

To express $-10$ in polar form, begin by sketching the graph of $-10$ in the complex plane, then find the vector with direction $\theta$ and magnitude $r$ that corresponds to the number in rectangular form.

Remember that $r = \sqrt{x^2 + y^2}$ and $\tan \theta = \frac{y}{x}$.

In polar form, $-10 = 10 \cis 180^\circ$.

Now, take the fourth roots. These are the numbers of the form $a + bi$, where $(a + bi)^4$ is equal to $10 \cis 180^\circ$. Suppose this complex number is $r \cos \alpha + i \sin \alpha$. Then you want the fourth power to be equal to $-10$, or $10 \cis 180^\circ$.

$$r^4(\cos 4 \alpha + i \sin 4 \alpha) = 10(\cos 180^\circ + i \sin 180^\circ)$$

Satisfy part of this equation by letting $r^4 = 10$. Solve this for $r$.

$$r^4 = 10 \rightarrow r = \sqrt[4]{10}$$

Next, make $n \alpha$ an angle coterminal with $180^\circ$. Therefore, you must have $4 \alpha = 180^\circ + 360^\circ \cdot k$, where $k$ is any nonnegative integer less than $n$.

$$\alpha = 45^\circ + 90k$$

Let $k$ take on integer values, starting with 0.

If $k = 0$, then $\alpha = 45^\circ$.

If $k = 1$, then $\alpha = 135^\circ$.

If $k = 2$, then $\alpha = 225^\circ$, and if $k = 3$, then $\alpha = 315^\circ$.

Notice that if $k = 4$, then $\alpha = 180^\circ + 360^\circ$, which will be coterminal with $180^\circ$, so you would just be repeating solutions already found if higher values of $k$ were used.

The solutions will be $\sqrt[4]{10} \cis 45^\circ$, $\sqrt[4]{10} \cis 135^\circ$, $\sqrt[4]{10} \cis 225^\circ$, and $\sqrt[4]{10} \cis 315^\circ$.

Find and graph the cube roots of $216i$.

Begin by writing $216i$ in trigonometric form.

The modulus is $\sqrt{0^2 + 216^2}$. Simplify the radical.

In trigonometric form, the modulus is 216.
Find the smallest nonnegative value \( \theta \) for which \( 216 = 216 \cdot \sin \theta \).

The argument, written as a nonnegative degree measure, is \( 90^\circ \).

\[ 216i = 216(\cos 90^\circ + i \sin 90^\circ) \]

The number of cube roots of \( 216(\cos 90^\circ + i \sin 90^\circ) \) is 3.

The roots of a complex number with modulus \( r \) all have the same modulus \( \frac{1}{r^n} \).

In this case \( r = 216 \).

\[ \frac{1}{r^n} = \frac{1}{216^3} = 6 \]

The \( n \)th roots of a complex number are distinguished by \( n \) different arguments. Find these arguments by using the following formula.

\[ \alpha = \{ \theta + 360^\circ \cdot k \} \text{ over } n \text{ for } k = 0, 1, 2, \ldots, n - 1. \]

In this case, \( n = 3, \theta = 90^\circ, k = 0, 1, \) and \( 2. \)

When \( k = 0, \)

\[ \alpha = \frac{90^\circ + (360^\circ)(0)}{3} = 30^\circ. \]

This gives the complex root \( 6(\cos 30^\circ + i \sin 30^\circ) \).

When \( k = 1, \)

\[ \alpha = \frac{90^\circ + (360^\circ)(1)}{3} = 150^\circ. \]

This gives the complex root \( 6(\cos 150^\circ + i \sin 150^\circ) \).

When \( k = 2, \)

\[ \alpha = \frac{90^\circ + (360^\circ)(2)}{3} = 270^\circ. \]

This gives the complex root \( 6(\cos 270^\circ + i \sin 270^\circ) \).

These three roots can be written in rectangular form and plotted.

\[ P_1 = (5.196, 3.000), \quad P_2 = (-5.196, 3.000), \quad \text{and} \]

\[ P_3 = (0.000, -6.000) \]

If these are the correct roots, the points will lie on a circle with center at \((0,0)\) and radius 6 (the modulus of the roots). The three points will divide the circumference of the circle in thirds.
Find and graph the fourth roots of 256.

Listed below are 8 choices lettered from a – h. Choose the correct solution(s) by typing the corresponding letters separated by commas.

- a) $4(\cos 0^\circ + i \sin 0^\circ)$
- b) $4(\cos 90^\circ + i \sin 90^\circ)$
- c) $4(\cos 180^\circ + i \sin 180^\circ)$
- d) $4(\cos 270^\circ + i \sin 270^\circ)$
- e) $4(\cos 45^\circ + i \sin 45^\circ)$
- f) $4(\cos 135^\circ + i \sin 135^\circ)$
- g) $4(\cos 225^\circ + i \sin 225^\circ)$
- h) $4(\cos 315^\circ + i \sin 315^\circ)$

Begin by writing 256 in trigonometric form. In trigonometric form, the modulus is $\sqrt{(256)^2+(0)^2} = 256$.

To find the argument, find the smallest nonnegative value of $\theta$ for which $256 = 256 \cdot \cos \theta$.

The argument, written as a nonnegative degree measure, is $0^\circ$.

$256 = 256(\cos 0^\circ + i \sin 0^\circ)$

There are 4 fourth roots of $256(\cos 0^\circ + i \sin 0^\circ)$.

The roots of a complex number with modulus $r$ all have the same modulus $\frac{1}{r^{\frac{1}{n}}}$. In this case $r = 256$.

$$r^{\frac{1}{n}} = 256^{\frac{1}{4}} = 4$$

The $n$th roots of a complex number are distinguished by $n$ different arguments. Find these arguments by using the following formula.

$$a = \frac{\theta + 360^\circ \cdot k}{n} \text{ for } k = 0, 1, 2, \ldots, n - 1.$$
In this case, \( n = 4 \), \( \theta = 0^\circ \), \( k = 0, 1, 2 \) and 3.

When \( k = 0 \),
\[
\alpha = \frac{0^\circ + (360^\circ)(0)}{4} = 0^\circ.
\]
This gives the complex root \( 4(\cos 0^\circ + i \sin 0^\circ) \).

When \( k = 1 \),
\[
\alpha = \frac{0^\circ + (360^\circ)(1)}{4} = 90^\circ.
\]
This gives the complex root \( 4(\cos 90^\circ + i \sin 90^\circ) \).

Note that \( \alpha = 180^\circ \) and \( 270^\circ \) for \( k = 2 \) and 3 respectively. The final two complex roots are \( 4(\cos 180^\circ + i \sin 180^\circ) \) and \( 4(\cos 270^\circ + i \sin 270^\circ) \).

The graph of the solution set shows all solutions are spaced \( 90^\circ \) apart.