Oblique Triangles and The Law of Sines

Congruency and Oblique Triangles ■ Derivation of the Law of Sines ■ Solving SAA and ASA Triangles (Case 1) ■ Area of a Triangle

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Law of Sines

In any triangle ABC, with sides a, b, and c,

$$\frac{a}{\sin A}=\frac{b}{\sin B}, \quad \frac{a}{\sin A}=\frac{c}{\sin C}, \text{ and } \frac{b}{\sin B}=\frac{c}{\sin C}.$$

Compact form:

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

Alternative form for solving for an angle:

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

Note: When using the law of sines, a good strategy is to select an equation so that the unknown variable is in the numerator and all other variables are known.

Solve the triangle, if possible.

A = 23.9° C = 112.8° c = 33.3

The sum of the three angles is 180°.

Therefore, 23.9 + 112.8 + B = 180°. Solve for B gives B = 43.3°.

To find the length of side a, the formula $\frac{a}{\sin A}=\frac{c}{\sin C}$.

Substitute the given values into the formula.

$$\frac{a}{\sin 23.9^\circ} = \frac{33.3}{\sin 112.8^\circ}$$

Solve for a.

$$a=(\sin 23.9^\circ)\frac{33.3}{\sin 112.8^\circ}$$

a = 14.6 (rounded to nearest tenth)

To find the length of side b, use the formula $\frac{b}{\sin B}=\frac{c}{\sin C}$.

Substitute the values of B, C, and c into this formula

$$\frac{b}{\sin 43.3^\circ}=\frac{33.3}{\sin 112.8^\circ}$$

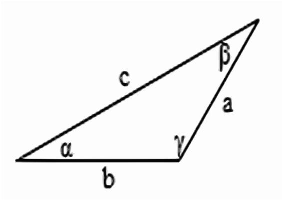
Solve for b.

$$b=(\sin 43.3^\circ)\frac{33.3}{\sin 112.8^\circ}$$

b = 24.8

Solve the triangle with the given parts.

$$\alpha=18.2^\circ, \gamma=117.5^\circ, c=59.3$$



The sum of the three angles of a triangle is 180° .

Therefore, 18.2 ° + 117.5 ° + β = 180 °. Solving, you get β = 44.3 °.

To find the length of side a, the following formula should be used:

$$\frac{a}{\sin\alpha}=\frac{c}{\sin\gamma}$$

Substitute into the formula.

$$\frac{a}{\sin 18.2^\circ}=\frac{59.3}{\sin 117.5^\circ}$$

$$a=\sin 18.2^\circ\cdot\frac{59.3}{\sin 117.5^\circ}$$ = 20.881 → to nearest tenth: a = 20.9

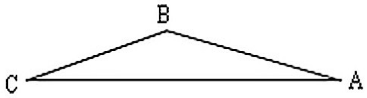
Find the length of side b.

$$\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$$

Substitute into this formula.

$$\frac{b}{\sin 44.3^\circ}=\frac{59.3}{\sin 117.5^\circ}$$ → to nearest tenth: b = 46.7

Solve the triangle with the given parts: angle C = 32.5 °, angle B = 113.7 °, and side AB = 54.4.



To find the length of side AC, use the following formula:

$$\frac{AC}{\sin B} = \frac{AB}{\sin C}$$

Substitute the given angles and the length of side AB into the formula:

$$\frac{AC}{\sin 113.7°} = \frac{54.4}{\sin 32.5°}$$

Solve for AC.

$$AC = (\sin 113.7°)\frac{54.4}{\sin 32.5°}$$

AC = 92.7 (rounded to nearest tenth)

Similarly, to find the length of side CB, use the formula

$$\frac{CB}{\sin A} = \frac{AB}{\sin C}.$$

Substitute the measures of angles A and C and the length of side AB into this formula.

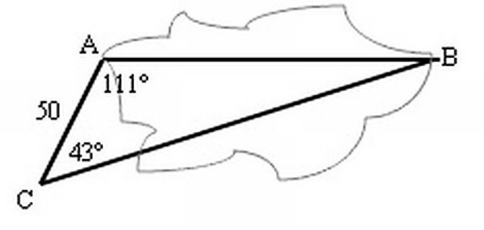
$$\frac{CB}{\sin 33.8°} = \frac{54.4}{\sin 32.5°}$$

Solve for CB.

$$CB = (\sin 33.8°)\frac{54.4}{\sin 32.5°}$$

CB ≈ 56.3 (rounded to nearest tenth)

Points A and B are on opposite sides of a lunar cater. Point C is 50 meters from point A. The measure of ∠ BAC is 111 ° and the measure of ∠ ACB is 43 °. What is the width of the crater?



To solve the problem, notice that you are looking for the length of the line joining point A to point B. It will be useful to draw a picture and label all the parts that you know.

Since you know the values of two angles and the included side, you can solve the triangle.

First, find the unknown angle.

The measure of $\angle$ ABC is 26 °.

Now, use the law of sines. Use the fact that $\frac{b}{\sin(B)}=\frac{c}{\sin(C)}$.

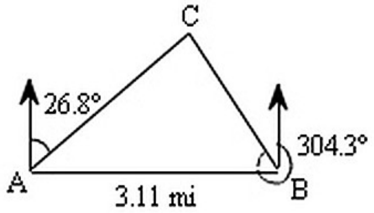
$$\frac{50}{\sin(26°)}=\frac{c}{\sin(43°)}$$

$$c=\frac{50\cdot\sin 43°}{\sin 26°}$$

c = 77.79 (rounded to nearest hundredth)

So, the length of side c, which is the width of the rater, is approximately 77.79 meters.

Radio direction finders are placed at points A and B, which are 3.11 mi apart on an east-west line, with A west of B. The transmitter has bearings 26.8 ° from A and 304.3 ° from B. Find the distance from A.



There are two radio direction finders on an east-west line, 3.11 miles apart, with a transmitter, which can be called C, marking bearing angles of 26.8 ° and 304.3 °.

Sketch a picture of this situation.

To find the distance AC, given two sides and the included angle, use the law of sines.

First, find the angles.

Angle CAB = 63.2 °.

The bearing from point B is 304.3 °, which means that the angle, measured clockwise from due north, is 304.3 °.

To find angle CBA, note that the bearing from B to A is 270 °.

Angle CBA = 34.3 °.

The sum of the three angles in a triangle is 180 °. Determine the measure of angle C.

63.2 ° + 34.3 ° + C = 180 °.

C = 82.5 °.

Now set up the law of sines to find side b.

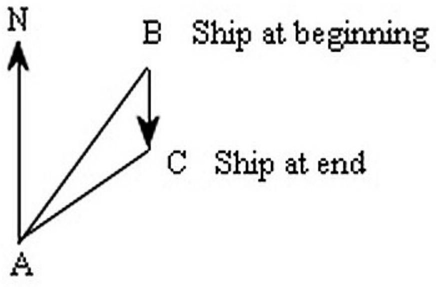
$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\frac{\sin 82.5°}{3.11} = \frac{\sin 34.3°}{b}$$

b = 1.77 mi (answer rounded to nearest hundredth)

The bearing of a ship from a lighthouse was found to be N 20 ° E. After the ship sailed 6.3 miles due south, the new bearing was N 30 ° E. Find the distance between the ship and the lighthouse at each location.

To solve this problem is helps to draw a picture. You have a lighthouse, which observes a ship at two different positions at two different bearings. Thus, you might draw the following picture:



Notice that the lighthouse, and the ship's two different positions form a triangle.

The measure of angle A of the triangle is A = 10 °

Other facts about the triangle can be determined.

In this triangle, B = 20 °, a = 6.3.

This information can be used to solve the triangle.

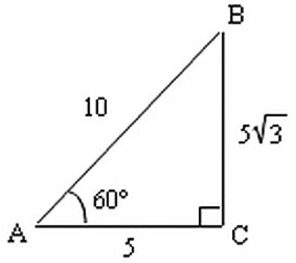
Angle C = 150 °.

Next, the law of sines can be used to find b and c.

b = 12.4
c = 18.1
(rounded to nearest tenth)

Notice that c is the distance between the lighthouse and the ship at the beginning, and b is the distance between the lighthouse and the ship at the end. Thus, the ship began 18.1 miles from the lighthouse, and ended 12.4 miles from the lighthouse.

Find the area of the triangle using the formula $A = \frac{1}{2}bh$, and verify that the formula $A = \frac{1}{2}bh \sin C$ give the same result.



To find the area using $A = \frac{1}{2}bh$, first determine the values of b and h.

For the given triangle, the base is b = 5. The height is $5\sqrt{3}$.

Substitute the values of b and h into the formula $A = \frac{1}{2}bh$, and simplify.

$$A = \frac{1}{2}(5)(5\sqrt{3}) = \frac{25\sqrt{3}}{2}$$

To use the formula $A = \frac{1}{2} bh \sin C$, determine the values of a, b, and C. Since a is the side opposite angle A and b is the side opposite angle B, the values of a and b are $a = 5\sqrt{3}$ and $b = 5$.

C = 90 °.

Substitute the values of a, b, and C into the formula $A = \frac{1}{2} bh \sin C$, and simplify.

$$A = \frac{1}{2}(5\sqrt{3})(5)\sin 90° = \frac{25\sqrt{3}}{2}$$

Therefore, the formula $A = \frac{1}{2} bh \sin C$ gives the same result as the formula $A = \frac{1}{2} bh$.

Find the area of the triangle.

a = 10.1 b = 5.9 C = 11.2 °

The formula for finding the area of a triangle can be written in three ways, but the correct formula here is:

$$A = \frac{1}{2} ab \sin C$$

Substitute.

$A = \frac{1}{2}(10.1)(5.9)\sin 11.2°$ = 5.8 (rounded to nearest tenth)

The Ambiguous Case of the Law of Sines

Description of the Ambiguous Case ■ Solving SSA Triangles (Case 2) ■ Analyzing Data for Possible Number of Triangles

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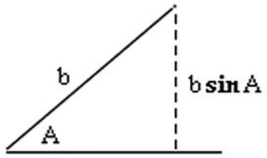
Determine the number of triangles ABC with the given parts.

a = 13, b = 9, A = 85 °



The number of possible triangles depends on the given angles and the relationship between the sides.

If angle A is obtuse, then the side opposite, a, must be the largest side in the triangle, and in particular, must be larger than the other given side, b. In this case, there will be only 1 possible triangle; otherwise, there are no possible triangles.

In the case of an acute angle, an important quantity is b sin A.

b sin A = 8.97 (rounded to nearest hundredth)

The picture shows the geometrical significance of b sin A.

In order for there to be any possible triangle, side a must be at least as long as b sin A. On the other hand, if side a is longer than b, one triangle is possible. Finally, if a is between b sin A and b, then two triangles are possible.

The number of possible triangles is 1.

Solve the triangle, if possible. Determine the number of possible solutions.

A = 84.2 ° a = 12.7 b = 9.4

You have been given 2 sides and an angle opposite one of them (SSA). There may be zero, one or two solutions in this case. If there are two solutions, it is an ambiguous case.

$$\frac{a}{\sin A}=\frac{b}{\sin B} \quad \rightarrow \quad \frac{12.7}{\sin 84.2\,°}=\frac{9.4}{\sin B} \quad \rightarrow \quad \sin B=\frac{9.4\sin 84.2\,°}{12.7}$$

angle B = 47.4 ° (rounded to nearest tenth)

In this case, there is 1 solution.

To find C, recall that the sum of all the angles in a triangle is 180 °.

$C=180\,°-A-B$ → C = 48.4 ° (rounded to nearest tenth)

Now, find the length of side c using the law of sines.

$$\frac{c}{\sin 48.4\,°}=\frac{12.7}{\sin 84.2\,°} \quad \rightarrow \quad c=\frac{12.7\sin 48.4\,°}{84.2\,°}$$

→ c = 9.5 (rounded to nearest tenth)

Solve the triangle, if possible. Determine the number of possible solutions.

A = 39.4 °   a = 3.5   c = 16.8

You have been given two sides and an angle opposite one of them (SSA). There may be zero, one or two solutions in this case. If there are two solutions, it is an ambiguous case.

First take the known measures of the triangle and use the law of sines to find angle C.

$$\frac{a}{\sin A}=\frac{c}{\sin C} \rightarrow \frac{3.5}{\sin 39.4°}=\frac{16.8}{\sin C} \rightarrow \sin C=\frac{16.8 \sin 39.4°}{3.5}$$

sin C = 3 (rounded to nearest tenth)

In this case, there are 0 solutions.

Determine the number of triangles with the given parts. If possible, find the angles of each triangle.

A = 41.9 °   a = 8.6   b = 10.1

You have been given two sides and an angle opposite one of them (SSA). There may be zero, one or two solutions in this case. If there are two solutions, it is an ambiguous case.

First, take the known measures of the triangle and use the law of sines to find angle B.

$$\frac{\sin A}{a}=\frac{\sin B}{b} \rightarrow \frac{\sin 41.9°}{8.6}=\frac{\sin B}{10.1} \rightarrow \sin B=\frac{10.1 \sin 41.9°}{8.6}$$

sin B = 0.78431 (rounded five decimal places)

There are two angles that have a sine of 0.78431 and are smaller than the supplement of angle A.

Therefore, there are 2 possible solutions.

If $B_1=128.3°$, then, $C_1=9.8°$. (rounded to nearest tenth)

For the second solution, $B_2=51.7°$ and $C_2=86.4°$.

Determine the number of triangles with the given parts. If possible, find the angles of each triangle.

A = 37.6 °, a = 3.6, c = 17.7

You have been given two sides and an angle not between them

(SSA), so this is an ambiguous case. There may be zero, one or two triangles with the given parts.

If $a \geq c$, then only one triangle is formed. However, 3.6 < 17.7, so you must find h (the altitude from B to the initial side of A) to test the possibilities.

Remember $\sin A = \frac{h}{c}$, where c = 17.7 and A = 37.6 °.

Thus h is 10.8. (rounded to nearest tenth)

Notice that a < h (since 3.6 < 10.8).

Since a < h, the number of triangles possible is 0.

Since there are 0 triangles with the given parts, you cannot solve for them.

Determine the number of triangles with the given parts. If possible, find the angles of each triangle.

B = 138.18 °   c = 7.6   b = 15.6

You have been given 2 sides and an angle opposite one of them (SSA). There may be 0, 1 or 2 solutions in this case. If there are 2 solutions, it is an ambiguous case.

Since B is more than 90 °, it is obtruse, there are only two possibilities:

If $b \leq c$, there are no solutions possible. If b > c, then one solution is possible.

In this case, there is 1 solution.

Now solve the triangle using the law of sines. First, find C.

$$\frac{\sin 138.18°}{15.6} = \frac{\sin C}{7.6} \quad \rightarrow \quad \sin C = 7.6 \cdot \frac{\sin 138.18°}{15.6} \quad \rightarrow$$

$$C = \sin^{-1}\left(\frac{7.6 \cdot \sin 138.18°}{15.6}\right)$$

The measure of angle C is 18.96 °. (rounded to nearest hundredth)

To find A, remember that all the angles of a triangle must add to 180°.

$$A = 180° - B - C. \quad \rightarrow \quad A = 180° - 138.18° - 18.96°$$

A = 22.86 ° (rounded to nearest hundredth)

Determine the number of triangles with the given parts. If possible, find the angles and sides of the triangle.

A = 42 ° 30'   a = 8.7 m   b = 10.3 m

You have been given two sides and an angle opposite one of them (SSA). There may be zero, one or two solutions in this case.

First, take the known measures of the triangle and use the law of sines to find angle B.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \rightarrow \frac{\sin 42.5°}{8.7} = \frac{\sin B}{10.3}$$

Solve for sin B.

$$\sin B = \frac{10.3 \sin 42.5°}{8.7} = 0.79984 \text{ (rounded to five decimal places)}$$

Use the inverse sine functions to find B.

$$B_1 = 53.1°$$

There are two angles between 0 ° and 180 ° that satisfy sin B = 0.79984. Supplementary angles have the same sine value. Another possible value of B is $B_2 = 126.9°$.

To see if $B_2 = 126.9°$ is a valid possibility add 126.9 ° to 42.5° . Since the sum is less than 180 °, $B_2 = 126.9°$ is a valid possibility. This is an ambiguous case.

Therefore, there are 2 possible solutions.

The Law of Cosines

Derivation of the Law of Cosines ■ Solving SAS and SSS Triangles (Cases 3 and 4) ■ Heron's Formula for the Area of a Triangle

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Assume triangle ABC has standard labeling and complete the statement below. Determine if AAS, ASA, SSA, SAS, or SSS is applicable, then decide if the law of sines or the law of cosines should be used as the first step to solve the triangle.

b, c, and B

Choose the appropriate representation of b, c, and B: SSA

The law of sines should be used here. The law of sines should be used unless three sides (SSS) or two sides and the angle (SAS) are given.

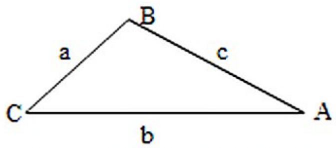
Assume triangle ABC has standard labeling and complete the statement below. Determine if AAS, ASA, SSA, SAS, or SSS is applicable, then decide if the law of sines or the law of cosines should be used as the first step to solve the triangle.

c, B, and b

Choose the appropriate representation of c, B, and b: SSA

The law of sines should be used here. The law of sines should be used unless three sides (SSS) or two sides and the angle (SAS) are given.

Solve the triangle if possible.



a = 9   c = 10   B = 98.7°

To solve for the length of side b, use the formula shown below.

$$b^2 = a^2 + c^2 - 2ac\cos B$$

Solve for b by substituting the given values into the formula.

$$b = \sqrt{9^2 + 10^2 - 2(9)(10)\cos 98.7°}$$

$$b = \sqrt{9^2 + 10^2 - (180)\cos 98.7°}$$ = 14.4 (rounded to nearest tenth)

Use the law of sines to solve for C.

The formula to use is: $\frac{\sin B}{b} = \frac{\sin C}{c}$

Substitute.

$$\frac{\sin 98.7°}{14.4} = \frac{\sin C}{10}$$

Solve for C.

$$C = \sin^{-1}\left(\frac{10 \sin 98.7°}{14.4}\right)$$

$C \approx 43.2°$

Use the measures of angles B and C to find the measure of angle A.

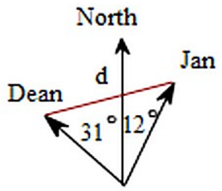
The sum of the three angles of any triangle is 180 °.

$98.7° + 43.2° + A = 180° \rightarrow A = 38.1°$

Jan and Dean started hiking from the same location at the same time. Jan hiked at 4 mph with a bearing of N12 °E, and Dean hiked at 4 mph with a bearing of N31 °W.

Find the distance between Jan and Dean after 2 hours.

To solve this problem, draw a picture and label the events.



In this picture, the side labeled d represents the distance between the hikers after 2 hours. To find d, you can use the law of cosines.

First, determine the lengths of the others sides of the triangle.

Jan hiked at 4 mph for 2 hours, so Jan traveled 8 miles.

Dean hiked 4 mph for 2 hours, so Dean traveled 8 miles.

To use the law of cosines, you must also determine the measure of the angle opposite side d.

The measure of this angle is 43 °.

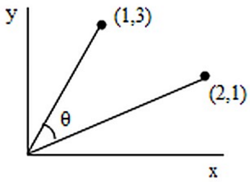
Using the law of cosines with the known information, you get the following.

$$d^2 = (8)^2 + (8)^2 - 2(8)(8)\cos 43° \rightarrow 64 + 64 - (128)(0.73135370)$$

$\rightarrow 34.387$

The distance between them is 5.9 miles.

Find the measure of the angle θ.



Use the law of cosines to find the included angle. This means find the three sides of the triangle, a, b, and c.

The length of side a, the side from the origin to (2,1), is 2.236068. (rounded to six decimal places)

The length of side b, the side from the origin to (1,3), is 3.162278, rounded to six decimal places.

The third side, c, runs from point (2,1) to the point (1,3). Use the distance formula to find the length of this side.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The third side has a length of 2.236068, rounded to six decimal places.

Now use the law of cosines and solve for the angle C, or θ.

$$c^2 = a^2 + b^2 - 2\text{ab}\cos C$$

$$(2.236068)^2 = (2.236068)^2 + (3.162278)^2 - 2(2.236068)(3.162278)$$

$$\rightarrow \quad 5 = 5 + 10 - 14.142136\cos C \quad \rightarrow -10 = -14.142136 \cos \text{C}$$

cos C = 0.707107

Use the inverse cosine function to find C, or θ: 45 °.