Inverse Circular Functions

Inverse Functions ■ Inverse Sine Function ■ Inverse Cosine Function ■ Inverse Tangent Function ■ Remaining Inverse Circular Functions ■ Inverse Function Values

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Domain of inverse = range of original function

Domain of inverse sin function: Same as sin: [–1,1]
Domain of inverse cos function: Same as cos: [–1,1]

Range of arcsin(x): [(–π/2),(π/2)]

Find the exact value, in radians, of the expression $\tan^{-1}(-\sqrt{3})$.

To find $\tan^{-1}(-\sqrt{3})$, recall that you are looking for an angle, α, so that tan(α) = $-\sqrt{3}$

Also, remember that the angle must lie in the interval $(-\frac{\pi}{2},\frac{\pi}{2})$.

The correct value for α in the interval $(-\frac{\pi}{2},\frac{\pi}{2})$ that makes $\tan\alpha=-\sqrt{3}$ is $-\frac{\pi}{3}$.

So, $\tan^{-1}(-\sqrt{3})=-\frac{\pi}{3}$ radians.

$$\csc^{-1}(-\frac{2}{\sqrt{2}}) = \sin^{-1}(-\frac{\sqrt{2}}{2}) = -45°(\frac{\pi}{180})$$

Find the approximate value, in radians, of the following expression, using a calculator.

$\cot^{-1}(-0.082)$ = 1.65 rad

Note: Since –0.082 is less than zero, use $\cot^{-1}(\alpha)=\tan^{-1}(\frac{1}{\alpha})+\pi$

Trigonometric Equations I

Solving by Linear Methods ■ Solving by Factoring ■ Solving by Quadratic Methods ■ Solving by Using Trigonometric Identities

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Find the approximate value, in radians, of the following expression, using a calculator.

$$2\cos x=\sqrt{2}$$

First, divide by 2 to solve for cos x.

$$\cos x=\frac{\sqrt{2}}{2}$$

To solve a problem like this, you should first find the angle of least positive measure whose cosine is $\frac{\sqrt{2}}{2}$. You can determine this by finding the inverse cosine of $\frac{\sqrt{2}}{2}$.

$$\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

If this angle is negative, you can change it to a positive angle with the same terminal side by adding 2π.

In this case, the angle is already positive.

Since the period of cosine is 2π, then any integral multiple of 2π can be added to this solution to obtain other solutions. So this equation is satisfied by $\frac{\pi}{4}+2\pi k$.

In general, there will be other angles, not of the form $\frac{\pi}{4}+2\pi k$, that also satisfy this equation.

In most cases, there will be at least two angles whose cosine is a given quantity. To locate another angle, notice where the arc of length $\frac{\pi}{4}$ terminates along the unit circle.

To find another angle whose cosine is equal to $\frac{\sqrt{2}}{2}$, find a second arc that terminates at a place with the same x–values as this arc.

A second arc that terminates with the same x–value is the arc of length $7\frac{\pi}{4}$.

Thus, x = $7\frac{\pi}{4}$ is a second solution, and all angles of the form $7\frac{\pi}{4}+2\pi k$ will also be solutions.

The solutions are therefore {x|x = $\frac{\pi}{4}+2\pi k$ } and {x|x = $7\frac{\pi}{4}+2\pi k$ }.

Solve the equation for solutions in the interval [0, 2π).

$\tan^2 x - 1 = 0$

First write down the equation: $\tan^2 x - 1 = 0$

Notice this is an equation that is quadratic in tan x. we can use either the quadratic formula or factoring to solve it for tan x.

Factoring: $(\tan x - \sqrt{1})(\tan x + \sqrt{1})$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad =$$

$$\tan x = \frac{-(0) \pm \sqrt{(0)^2 - 4(1)(-1)}}{2(1)} \quad \rightarrow \tan x = 1, -1$$

tan x = 1 when x = π/4, 5π/4
tan x = –1 when x = 3π/4, 7π/4

So, the solutions to $\tan^2 x - 1 = 0$ are π/4, 5π/4, 3π/4 and 7π/4.

Find all real numbers in the interval [0,2π) that satisfy the equation.

$7\cos^2 x + 15\cos x + 8 = 0$

This is a quadratic equation in cos x.

$$\cos x = -\frac{8}{7}, \cos x = -1$$

$$\cos^{-1}\left(-\frac{8}{7}\right) = N, \cos^{-1}(-1) = \pi$$

This solution set is {π}.

Solve the equation for exact solutions in the interval [0°, 360°). Use an algebraic method.

$(\cot\theta - \sqrt{3})(2\sin\theta + \sqrt{3}) = 0$

The equation is already factored and set equal to 0, so set each factor equal to 0 separately.

First part:
$\cot\theta - \sqrt{3} = 0 \rightarrow \cot\theta = \sqrt{3} \rightarrow \theta = \cot^{-1}(\sqrt{3})$

θ= 30°,210°

tan(30) = tan(210)

Find the other angle on the unit circle with the same x and y values as 30. This should be 210.

Second part:

$$2\sin\theta+\sqrt{3}=0 \;\rightarrow\; \sin\theta=-\frac{\sqrt{3}}{2} \;\rightarrow\; \theta=\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -60$$

$\theta = -60$ or $(360-60) = 300$

The y value at 300 is $-\frac{\sqrt{3}}{2}$

The y value at 240 is $-\frac{\sqrt{3}}{2}$

So, $\theta = 240, 300$

sin(240) = sin(300)

Angle on unit circle with negative x and y value: 240

So, $\theta = 30, 210, 240, 300°$

Solve with a grapher, finding all solutions in $[0,2\pi)$.

$-x + 5\sin(x) + 2 = 4 \rightarrow -x + 5\sin(x) - 2 = 0$

Trigonometric Equations II

Equations with Half–Angles ■ Equations with Multiple Angles

Solve the equation for solutions in the interval $[0,2\pi)$. Use algebraic methods and give exact values. Support your solution graphically.

$$\cos 2x=\frac{\sqrt{2}}{2}$$

To solve this equation for x, begin by noting that $\cos 2x=\frac{\sqrt{2}}{2}$. First, observe that since $0\leq x<2\pi$, then $0\leq 2x<4\pi$. Thus, begin by finding all the solutions to $\cos 2x=\frac{\sqrt{2}}{2}$ in the interval $[0, 4\pi)$.

$\cos\left(\frac{\sqrt{2}}{2}\right)$ = 45 (or, $\pi/4$), 315 (or, $7\pi/4$)

Add 2π to both $\pi/4$ and $7\pi/4$ for the four answers to get

$2x = \pi/4, 7\pi/4, 9\pi/4, 15\pi/4$

Divide by 2 to solve for x.
x = π/8, 7π/8, 9π/8, 15π/8

Solve the equation for solutions in the interval [0,2π). Use algebraic methods and give exact values. Support your solutions graphically.

$$\sin\left(\frac{x}{2}\right)=\sqrt{2}-\sin\left(\frac{x}{2}\right)$$

First note that the interval is $0 \le x < 2\pi$. This means $0 \le \frac{x}{2} < \pi$

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Rearrange the equation so that the sine terms are all on one side.

2 sin (x/2) = $\sqrt{2}$

sin (x/2) = $\frac{(\sqrt{2})}{2}$

The solutions in the interval 0 <= x/2 <π are:
x/2 = π/4, 3π/4 (After finding π/4, note that it is 45 °rees. 180 minus 45 is 135. 135° equals 3π/4rad.)

Solve for x: x = π/2, 3π/2

Solve the equation over the interval [0,2π), then support your solutions graphically.

cos 2x + cos x = 0

Write the equation so that the trig functions are in terms of the same thing. This means you will want to change cos 2x to an expression of sine and cosine of x.

cos 2x can be written as $2(\cos^2 x)-1$.

Note: cos 2x is not equal to 2 cos x.

Replace cos 2x with $(2\cos^2 x)-1$

$$(2\cos^2 x)-1+\cos x=0$$

Notice that this equation is quadratic, so we can use either the quadratic formula or factoring to solve it for cos x. We can factor it.

2 cos^2 x + cos x – 1 = (cos x + 1)(2 cos x – 1)

To solve, we can set either cos x + 1 or 2 cos x – 1 equal to 0.

cos x + 1 = 0 → x = –1
2 cos x – 1 = 0 → x = 1/2

In the required interval, x = π, π/3, 5π/3

Solve the equation over the interval [0, 2π), and support your solution graphically.

$$\sin x \cos x = \frac{\sqrt{2}}{4}$$

To solve a trigonometric equation, it is convenient to write it in terms of a single trigonometric function. To do this, refer to the identities that you have.

The identity in which sin x cos x appear is sin 2x.

Note: sin 2x is not equal to 2 sin x.

Since sin 2x = 2 sin x cos x, rewrite the equation to contain 2 sin x cos x by multiplying both sides by 2, then apply the identity.

Since sin 2x = 2 sin x cos x, rewrite the equation to contain 2 sin x cos x by multiplying both sides by 2, then apply the identity.

$$\sin x \cos x = \frac{(\sqrt{2})}{4}$$

$$2 \sin x \cos x = 2\left(\frac{\sqrt{2}}{4}\right)$$

$$2 \sin x \cos x = \frac{(\sqrt{2})}{2}$$

$$\sin 2x = \frac{(\sqrt{2})}{2}$$

Since the equation involves a function of 2x, look for values $0 \le x < 2\pi$, or $0 \le 2x < 4\pi$.

The values of 2x, in the interval [0, 4π), for which sin 2x = $\frac{(\sqrt{2})}{2}$ are 2x = π/4, 3π/4, 9π/4, 11π/4

So, x = π/8, 3π/8, 9π/8, 11π/8

So, graph Y_1 = sin x cos x, and $Y_2 = \frac{(\sqrt{2})}{4}$

The solutions correspond to the points where the two graphs intersect.

Find all values in [0,360°) that satisfy the given equation.

$$-6\left(\sqrt{2}\right)\cos 2\alpha - 6 = 0$$

To solve, first write the equation in the form cos a = b, then take the inverse cosine of both sides. Rearrange the terms so that the equation is in this form.

$$\cos 2\alpha = -\frac{\left(\sqrt{2}\right)}{2}$$

Next, find the values of 2α between 0° and 360 ° that satisfy the equation.

The angles between 0° and 360° which have a cosine of $-\frac{\left(\sqrt{2}\right)}{2}$ are 135,225

$$\cos 2\alpha \quad = \quad -\frac{\left(\sqrt{2}\right)}{2}$$

2α = 135 + k360° or 2α = k360°

Solve the equation on the left for α.

α = 67.5 + k(180)

To get all the solutions in the interval [0,360°), note that k is any integer. If k=0, then α = 67.5. If k=1, then α = 247.5.

The second equation, solved for α, gives us α = 112.5 + k 180°.

This gives us the solutions α = 112.5, 292.5

So, the solutions are α = 67.5, 247.5, 112.5, 292.5°

Equations Involving Inverse Trigonometric Functions
Solving for *x* in Terms of *y* Using Inverse Functions ■ Solving Inverse Trigonometric Equations

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Solve the equation for x.

5y = sec 4x

To solve for x, get 4x by itself. Apply the arcsec function to both sides.

4x = arcsec 5y

Divide by 4.

x = 1/4 arcsec 5y

Solve the equation for x.

y = tan x – 8

Begin by getting tan x by itself. Add 8 to both sides.

y + 8 = tan x

Now apply the arctan function to both sides.

x = arctan (y+8)

Solve the equation for an exact solution.

$$\frac{6}{5}\cos^{-1}\left(\frac{y}{8}\right)=\pi$$

To solve this equation, begin by multiplying both sides by 5/6 to isolate $\cos^{-1}$.

$$\cos^{-1}\left(\frac{y}{8}\right)=\frac{5}{6}\pi$$

Next, use the definition of $\cos^{-1}x$ to simplify this equation.

If $y=\cos^{-1}x$, the cos y = x.

Since $\cos^{-1}(y-8)$ = $\frac{5}{6}\pi,\cos\left(\frac{5}{6}\right)\pi$ = $\frac{y}{8}$

= $-\frac{(\sqrt{3})}{2}=\frac{y}{8}$

Multiply both sides by 8.

$$-8\frac{\sqrt{3}}{2}=y$$

Reduce.

$y = -4\sqrt{3}$

Solve the equation for an exact solution.

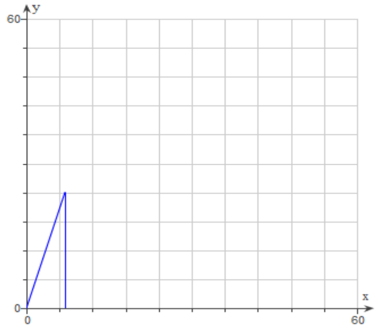
arcsin x = arctan 24/7

Use the definition of arcsin to rewrite this equation.

x = sin (arctan 24/7)

If you let u = arctan 24/7, then $-\pi/2 < u < \pi/2$ by definition, and x = sin u.

Sketch a triangle with an angle u that has tan u = 24/7.



Since the side adjacent to the angle u is 7, and the side opposite the angle u is 24, you can also find the hypotenuse.

The hypotenuse is h = 25

You can then find x.

x = sin u

x = 24/25

Solve the equation for an exact solution.

$$\sin^{-1} x - \tan^{-1} 3 = -\frac{\pi}{3}$$

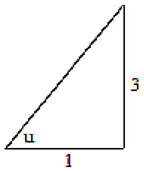
First, isolate $\sin^{-1} x$ on one side by adding $\tan^{-1} 3$ to both sides.

$$\sin^{-1} x = -\frac{\pi}{3} + \tan^{-1} 3$$

Take the sine of both sides.

$$\sin\left(-\frac{\pi}{3}+\tan^{-1}3\right)$$

Let $u=\tan^{1}3$. By definition, $-\frac{\pi}{2}<u<\frac{\pi}{2}$. Draw a triangle where tan u = 3.



Determine the hypotenuse of this triangle.

h = $\sqrt{10}$

Use the sine of a sum formula to find sin (–π/3 + u).

$$\sin\left(-\frac{\pi}{3}+u\right)=\sin\left(-\frac{\pi}{3}\right)\cos u+\cos\left(-\frac{\pi}{3}\right)\sin u=-\frac{\sqrt{3}}{2}\cos u+\frac{1}{2}\sin u$$

To find the exact value, determine the sine and cosine of u.

cos u = $\frac{1}{(\sqrt{10})}$, sin u = $\frac{3}{(\sqrt{10})}$

Substitute these values in an simplify.

$$x=\sin\left(-\frac{\pi}{3}+u\right)=\left(-\frac{\sqrt{3}}{2}\right)\frac{1}{\sqrt{10}}+\left(\frac{1}{2}\right)\frac{3}{\sqrt{10}}$$

$$x=\frac{3\sqrt{10}-\sqrt{30}}{20}$$