Related Rates Problem Set

(Revised March 8, 2011)

1. If $z^2 = x^2 + y^2$, $dx/dt = 2$, and $dy/dt = 3$, find dz/dt when $x = 5$ and $y = 12$. Assume that $z \geq 0$.

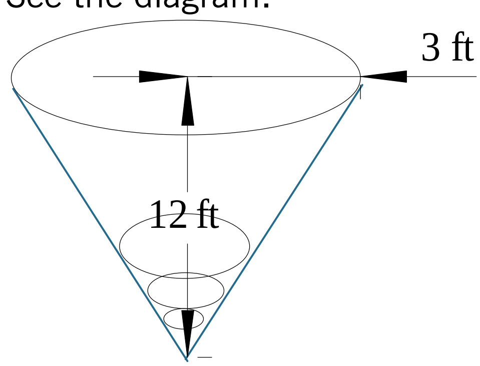
2. A particle moves along the curve $y = \sqrt{1 + x^3}$. As it reaches the point $(2, 3)$, the y-coordinate is increasing at a rate of 4 cm/sec. How fast is the x-coordinate of the point changing at this instant?

3. Suppose $PV = k$ where k is a constant and $\frac{dP}{dt} = 2$ when $P = 4$ and $V = 5$. Find $\frac{dV}{dt}$.

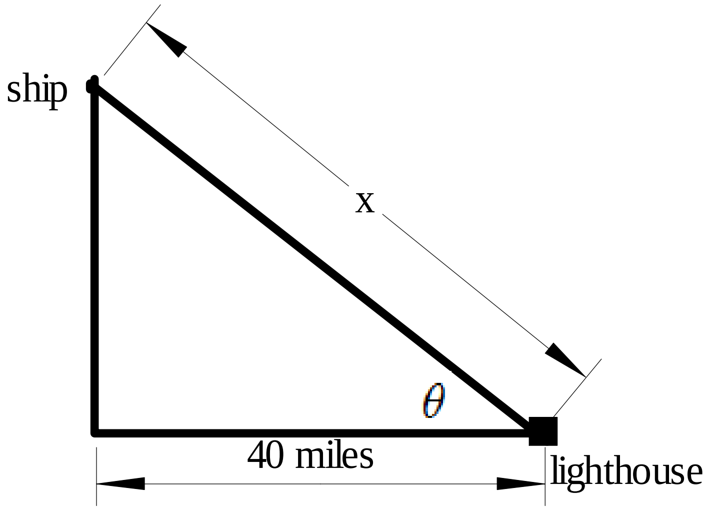
4. Two cars start moving from the same point. One travels south at 60 m/hr and the other travels west at 25 m/hr. At what rate is the distance between the cars increasing two hours later?

5. A plane flying horizontally at an altitude of 1 mile and a speed of 500 mph passes over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 miles away from the station.

6. A conical water tower has a height of 12 ft and a radius of 3 ft. Water is pumped into the tank at a rate of 4 ft^3/min. How fast is the water level rising when the water level is 6 ft? See the diagram.

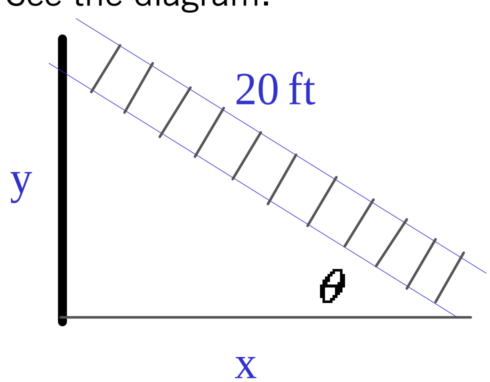


7. A ship is 40 miles west of a lighthouse. The ship is heading north at a rate such that the angle θ, shown in the diagram below, is changing at a constant rate of 0.7 radians per hour. At what rate is the distance x between the ship and the lighthouse changing when $\theta = 0.4$ radians?



8. At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/hr and ship B is sailing north at 25 km/hr. How fast is the distance between the ships changing at 4:00 P.M.?

9. An 20 ft long ladder is leaning against a wall. The bottom of the ladder is sliding away from the wall at a rate of 2.5 ft/sec. See the diagram.



a) How fast is the top of the ladder sliding down the wall when $x = 12$ ft. Note that this rate is $|dy/dt|$.

b) How fast is the angle θ changing when $x = 12$ ft?

c) How fast is the area of the triangle changing when $x = 12$ ft?

Solutions

1. If $z^2 = x^2 + y^2$ then
$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \quad \Rightarrow$$
$$z' = \frac{xdx/dt + ydy/dt}{z} = \frac{(5)(2) + (12)(3)}{\sqrt{5^2 + 12^2}} = \frac{46}{13}\,.$$

2. If $y = \sqrt{1 + x^3}$ then $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt} \quad \Rightarrow$

$$\frac{dy}{dt} = \frac{1}{2}\left(1 + x^3\right)^{-1/2}\left(3x^2\right)\frac{dx}{dt} \quad \Rightarrow$$

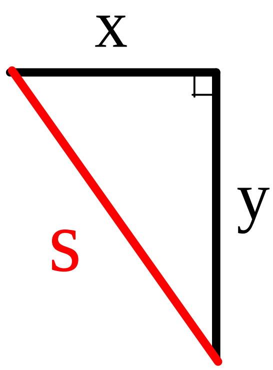
$$4 = \frac{1}{2(1 + 2^3)^{1/2}}\left(3 \cdot 2^2\right)\frac{dx}{dt} \quad \Rightarrow$$

$$4 = \frac{12}{6}\frac{dx}{dt} \quad \Rightarrow \quad \frac{dx}{dt} = 2\,\text{cm/sec}\,.$$

3. If $PV = k$ then $\dfrac{dPV}{dt} = \dfrac{dk}{dt} \quad \Rightarrow$

$$\frac{dP}{dt}V + P\frac{dV}{dt} = 0 \quad \Rightarrow \quad 2(5) + 4\frac{dV}{dt} = 0 \quad \Rightarrow$$
$$\frac{dV}{dt} = \; -\frac{10}{4} = -\frac{5}{2}\,.$$

4. Look at the situation t hrs later, where t is an arbitrary time.
The 1st car has moved south (say a distance y) and the 2nd car has moved west (say a distance x). We have the triangle below.



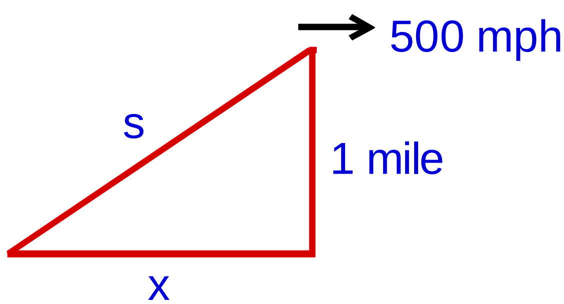
$x^2 + y^2 = s^2$ ← differentiate both sides with respect to t

$$2xx' + 2yy' = 2ss' \quad \Rightarrow \quad s' = \frac{xx' + yy'}{s}$$

where $x' = 25$ and $y' = 60$ and s' is the unknown in the problem. Note that 2 hours later, $x = 50$ m, $y = 120$ m, and $s = \sqrt{50^2 + 120^2} = 130$ m. Therefore

$$s' = \frac{(50)(25) + (120)(60)}{130} = 65 \text{ m/hr.}$$

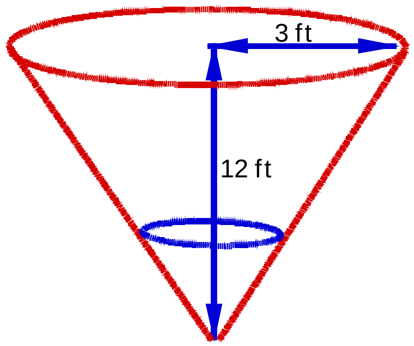
5.

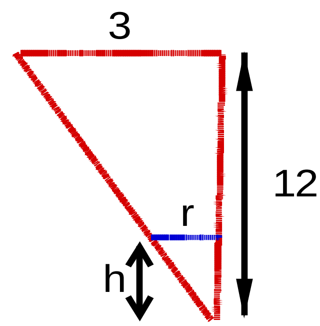


$$s^2 = x^2 + 1^2 \quad \Rightarrow \quad 2ss' = 2xx' \quad \Rightarrow \quad s' = \frac{xx'}{s} = \frac{(500)\left(\sqrt{4-1}\right)}{2}$$

$250\sqrt{3} = 433$ mph.

6. Let h be the height of the water lever and let r be the radius of the cone of water.





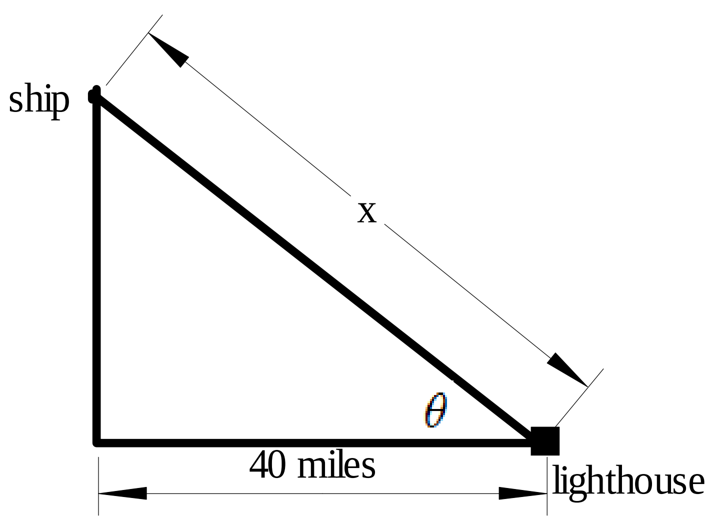
Because of similar triangles, $\frac{12}{3} = \frac{h}{r} \quad \Rightarrow \; h = 4r \quad \Rightarrow \; r = \frac{h}{4}$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h = \frac{\pi}{48}h^3$$

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt} = \frac{1}{16}\pi h^2\frac{dh}{dt} = 4 \quad \Rightarrow \quad \frac{dh}{dt} = \frac{64}{\pi h^2} = \frac{64}{\pi(6)^2} =$$

$\frac{64}{36\pi} = \frac{16}{9\pi}$ ft/min .

7. We want $\frac{dx}{dt}$ when $\frac{d\theta}{dt} = 0.7\,\text{radians/hr}$.



Look at the drawing and find an equation involving x and θ.

One equation is $cos\,\theta = \dfrac{40}{x} \;\Rightarrow\; xcos\,\theta = 40 \;\Rightarrow$

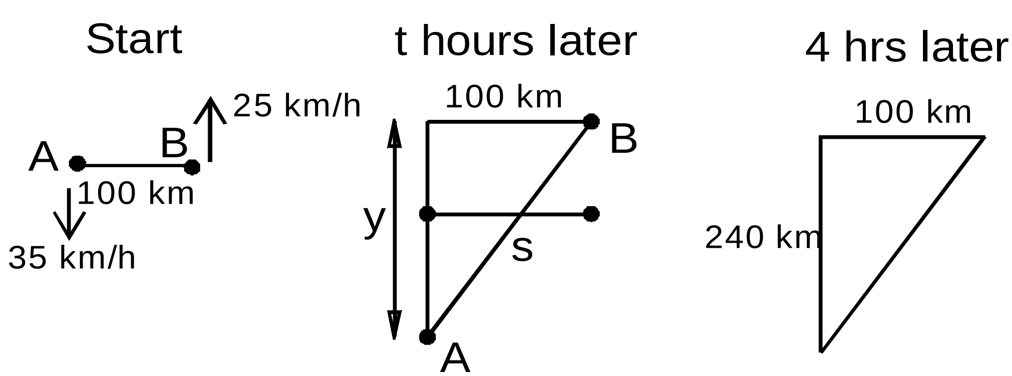
$$x = \frac{40}{cos\,\theta} \;\Rightarrow\; x = 40sec\,\theta\,.$$

Now differentiate both sides wrt t.

$$\frac{d}{dt}x = \frac{d}{dt}(40\,sec\,\theta) \;\Rightarrow\; x' = 40\,sec\,\theta\,tan\,\theta\cdot\theta' \;\Rightarrow$$

$$\frac{dx}{dt} = 40\,sec(0.4)tan(0.4)\cdot(0.7) = \mathbf{12.85\ mph}\,.$$

8.



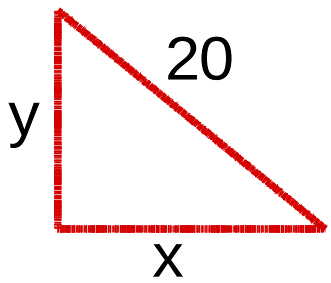
$$s^2 = y^2 + 100^2$$
$$2ss' = 2yy' \;\Rightarrow\; s' = \frac{yy'}{s}\,.$$

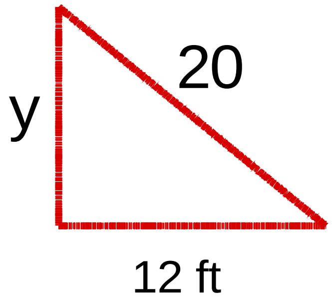
4 hrs later $y = 4(25) + 4(35) = 100 + 140 = 240\,km$

$y' = 25 + 35 = 60\,km/hr$

$$s' = \frac{(240)(60)}{\sqrt{240^2 + 100^2}} = 55.385\,km/hr$$

9.





a) $x^2 + y^2 = 400 \quad \Rightarrow \quad 2xx' + 2yy' = 0 \quad \Rightarrow$

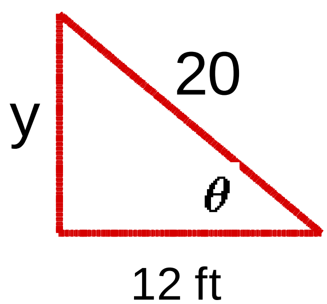
$$y' = \frac{-xx'}{y}.$$

When $x = 12$, $y^2 + 144 = 400 \quad \Rightarrow \quad y = \sqrt{400 - 144} = 16$.

$$y' = \frac{-(12)(2.5)}{16} = -1.875 \text{ ft/sec}$$

Therefore the top of the ladder is sliding down at a speed of 1.875 ft/sec.

b)



$$sin\,\theta = \frac{y}{20} \quad \Rightarrow \quad y = 20\,sin\,\theta \quad \Rightarrow \quad \frac{dy}{dt} = 20\,cos\,\theta\,\frac{d\theta}{dt}$$

$$\Rightarrow \quad \frac{d\theta}{dt} = \frac{y'}{20\,cos\,\theta} = \frac{-1.875}{20(12/20)} = -0.156 \text{ radians/sec}$$

c) $A = \frac{1}{2}xy \quad \Rightarrow \quad \frac{dA}{dt} = \frac{1}{2}\frac{dx}{dt}y + \frac{1}{2}x\frac{dy}{dt} =$

$$\frac{1}{2}(2.5)(16) + \frac{1}{2}(12)(-1.875) = 8.75 \text{ ft}^2\text{/sec}.$$